Exercise 1

Use power series to solve the differential equation.

$$y' - y = 0$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substitute these formulas into the ODE.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Make the substitution $n = k + 1$ in the first series and the substitution $n = k$ in the second series.

$$\sum_{k+1=1}^{\infty} (k+1) a_{k+1} x^{(k+1)-1} - \sum_{k=0}^{\infty} a_k x^k = 0$$

Simplify the first sum.

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Now that all the sums start from $k = 0$ and have x^k in the summand, they can be combined.

$$\sum_{k=0}^{\infty} \left[(k+1) a_{k+1} x^k - a_k x^k \right] = 0$$

Simplify the summand.

$$\sum_{k=0}^{\infty} \left[(k+1) a_{k+1} - a_k \right] x^k = 0$$

Since x^k isn't zero, the quantity in square brackets must be zero.

$$(k+1) a_{k+1} - a_k = 0$$

Solve for a_{k+1}.

$$a_{k+1} = \frac{1}{k+1} a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$k = 0: \quad a_1 = \frac{1}{0+1}a_0 = \frac{1}{1}a_0$$

$$k = 1: \quad a_2 = \frac{1}{1+1}a_1 = \frac{1}{2}\left(\frac{1}{1}a_0\right) = \frac{1 \cdot 1}{2 \cdot 1}a_0$$

$$k = 2: \quad a_3 = \frac{1}{2+1}a_2 = \frac{1}{3}\left(\frac{1 \cdot 1}{2 \cdot 1}a_0\right) = \frac{1 \cdot 1 \cdot 1}{3 \cdot 2 \cdot 1}a_0$$

$$\vdots$$

The general formula is

$$a_m = \frac{1}{m!}a_0.$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} \left(\frac{1}{m!}a_0\right) x^m \\ &= a_0 \sum_{m=0}^{\infty} \frac{x^m}{m!} \\ &= a_0 e^x, \end{aligned}$$

where a_0 is an arbitrary constant.